THE

GROUND-WORK OF NUMBER

A. S. ROSE

AND

S. E. LANG

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GROUND-WORK OF NUMBER

A MANUAL

—FOR—

THE USE OF PRIMARY TEACHERS.

BY

A. S. ROSE AND S. E. LANG,
INSPECTORS OF SCHOOLS, MANITOBA.

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CONTENTS.

CHAPTER I.

INTRODUCTION.

CHAPTER II.

RETROSPECT.

CHAPTER III.

THEORETICAL.

CHAPTER IV.

PRACTICAL.

CHAPTER V.

EXERCISES.

CONTENTS.

CHAPTER I.

INTRODUCTION.

CHAPTER II.

RETROSPECT.

CHAPTER III.

THEORETICAL.

PREFACE.

In this manual an attempt has been made to explain the nature of number and to set forth the conditions of clear thinking, with special application to the study of arithmetic. The main propositions are that number is a relation based upon the idea of time and not that of space, and that all thought begins in analysis. From the former of these, certain inferences are drawn which have an important bearing on the teaching of the subject. It will be seen that in this view of the case all the methods which deal with sense-perception of space as applied to arithmetic, that in the scientific study of number the use of objects is not only unnecessary, but also inconvenient, and even destructive of clear numerical ideas; that no true idea of a fraction can be formed from a consideration of the spatial relations of objects. The second proposition furnishes us with a test for the validity of all methods of procedure in the acquisition of knowledge.

THE GROUND-WORK OF NUMBER.

CHAPTER I.

INTRODUCTION.

1. A Science is a sum or body of knowledge which has been arranged according to certain definite rules. We may be in possession of much knowledge to which the term scientific can hardly be applied. Science is systematic knowledge. That a triangle whose sides are three, four, and five feet, respectively, always contains a right angle is a fact that is well known, and, even as an isolated fact, well worth knowing; but such knowledge is not scientific knowledge until the fact is thought of in connection with the forty-seventh proposition of Euclid.

Scientific procedure is of two kinds. Facts may be observed and classified according to the rules of inductive method; or, if the nature of the subject matter will admit, new truths may be discovered without a new set of observations for each case, and here the procedure is said to be

deductive. By observation is meant, in the study of the sciences of nature, the exercise of the powers of sense perception, with a view of discovering similarities among objects; more particularly, in the case of botany, similarities among plants. Classification signifies the arrangement of things—plants, for example—in groups or classes, on the basis of observed similarities. A theory may be advanced to explain the facts of a science. Its value as a theory depends on the possibility of explaining all the facts. Every newly discovered fact is a test of the validity of the theory. If it fails to explain the new fact, a new provisional hypothesis must be sought for. If, on the other hand, the explanation is adequate to the test, it is thereby confirmed.

The important feature in the inductive process is the accurate observation of facts. Inductive logic concerns itself with the harmony between the observed facts and the convenient general expressions known as theories. In those cases, on the other hand, in which a proposition depends, not upon the observation of facts, but upon its harmony or agreement with some previously established proposition, the procedure is said to

be deductive. Mathematical reasoning is of the latter kind. It does not seek to verify its conclusions by an appeal to observation. It is independent of such support. It relies wholly upon the validity of the primary truths upon which it is based.

2. Arithmetic is that branch of mathematical science which deals with numbers. Number is a relation or way of thinking by which order, coherence, unity, is given to our experience. Relation* is an act of mind without which no

* "Relation is to us such a familiar fact that we are apt to forget that it involves all the mystery, if it be a mystery, of the existence of the many in one . . . . . . A plurality of things cannot of themselves unite in one relation, nor can a single thing of itself bring itself into a multitude of relations. It is true, as we have said, that the single things are nothing except as determined by relations which are the negation of their singleness, but they do not therefore cease to be single things. Their common being is not something with which their several existences disappear. On the contrary, if they did not survive in their singleness, there could be no relation between them, nothing but a blank, featureless, identity. There must, then, be something other than the manifold things themselves, which combines them without effacing their severalty. With such a combining agency we are familiar as our intelligence. It is through it that the sensation of the present moment takes a character from comparison with the sensation of a moment ago, and that the occurrence, consisting in the transition from one to the other, is presented to us. It is essential . . . . that one should not be fused with the other; that the distinct being of each should be maintained. On the other hand, in the relation to which their distinctness is thus necessary, they are at the same time united. But if it were not for the action of something which is not either of them or both together, there would be no alternative between their separateness and their fusion."

—*Green: Prol. to Ethics, 28-29.*

thinking ever takes place. Thought, as has often been pointed out, consists in the establishment of relations. In thinking we organize into systematic relations that which would otherwise be separate and discrete in our experience; we unite ideas in thought because they appear alike, or keep them apart because they seem different; the idea of causation is the bond or basis of relation between certain ideas; we relate objects and events in space and time—these are examples of the various ways in which thought introduces system into the facts of our experience. Thus we see that to think is to order and arrange the materials of thought into unities of various kinds. But thought does not stop with the mere act of relation. By a process of abstraction we may become conscious of the relations themselves: abstracting from the ideas related, and fixing our attention upon the connection between them, we reach such ideas of relation as resemblance, difference, causation, quantity, and number. All ideas of relation, then, are due to the activity of the thought power, and number is a product of one of the phases of this mental activity.

3. The science of number is to be distinguished from that of form. We have seen that the action of the mind in giving order and coherence to the impressions of sense, gives rise to various ideas of relation. The science of geometry is made possible by the existence of one of these products of mental action in relating phenomena in the form of space. Arithmetic deals with number, geometry with form. In numbering, the mind organizes the data of sense in a time relation. It matters not which of the sense organs supplies the data, the organization of the data is in every case performed in the same way. The same form or bond,—that of time,—unites the impressions of sense in the act which gives rise to this product; and into this product the element of space does not enter. No analysis of the idea of space can ever bring forth the idea of number. Objects, in so far as they are conditioned by the relation of space, are co-existent. But they can be numbered only in so far as they can be thought of as discrete, *i.e.*, as successive. It will be seen that the possibility of numbering objects in

space depends upon our considering them in the time relation.*

* "If you wheel about a burning coal with rapidity, it will present to the senses an image of a circle of fire; nor will there be any interval of time betwixt two revolutions; meerly because 'tis impossible for our perceptions to succeed each other with the same rapidity, that motion can be communicated to external objects. Wherever we have no successive perceptions, we have no notion of time, even though there be a real succession in the objects."

—*Hume: Treatise on Human Nature, Part 2, Sec. 3.*

"I have already observed that time, in a strict sense, implies succession, and that when we apply its idea to any unchangeable object, 'tis only by a fiction of the imagination by which the unchangeable object is supposed to participate of the changes of the co-existent objects, and in particular of that of our perceptions. The fiction of the imagination almost universally takes place; and 'tis by means of it that a single object placed before us, and survey'd for any time without our discovering in it any interruption or variation, is able to give us a notion of identity. For when we consider any two points of this time, we may place them in different lights: we may either survey them at the very same instant, in which case they give us the idea of number, both by themselves and by the object: or, on the other hand, we may trace the succession of time, by a like succession of ideas, and conceiving first one moment, along with the object then existent, imagine afterwards a change in the time without any variation or interruption in the object; in which case it gives us the idea of unity."

—*Hume: Treatise on Human Nature, Part 4, Sec. 2.*

"Kant concludes his theory of the *a priori* forms of sense by noticing that the two forms are not altogether independent of each other. For if we compare them we see that the space is a determination of objects as such, while time has to do with states, and principally with the states of the subject as representing and perceiving objects. Hence, time stands between the conceptions of the understanding, and the perceptions of sense."

—*Caird: The Critical Philosophy of Kant, Ch. 5.*

4. The idea of number differs both in kind and origin from ideas of sound, touch, colour, and others which are due to the sense organs.* The senses furnish us with knowledge of the qualities of objects. Number is not a quality discernible in objects by means of any actual or possible sense organ. A very little reflection is sufficient to show that number is not due to the perceptive power. If it be held that the idea is in any way derived from sensation, we are entitled to ask from which department of sense, or from what combination of sensations, we get the idea. It is not due to the sense of sight, since colour is all that we get from that source; it is not due to hearing, or it would be a sound; and so of the others, touch, taste, and smell. The most delicate susceptibility to colour, sound, etc., is compatible with vague ideas of number. A person may be trained to appreciate slight differences in the various aspects,—inten-

* "No one is more emphatic than Locke in opposing what is real to what we "make for ourselves," the work of nature to the work of the mind. Simple ideas or sensations we certainly do not "make for ourselves." But relations are neither simple ideas nor their material archetypes. They therefore, as Locke explicitly holds, fall under the head of the work of the mind."

—*Green: Prol. to Ethics*, § 20.

sity, extent, duration,—of a given sensation. This power of discrimination may be developed to a very high degree. But such exercises cannot give the observer any ideas of number. Nor, on the other hand, is the study of number suited to the development of the powers of observation. The power to form correct ideas of numerical relation is not in the smallest degree dependent upon the extent of one's training in distinguishing shades of colour, or exactly estimating distances, or noticing differences in pitch of tones.

5. Ideas of numerical relation are not generalizations from experience. The term generalization indicates that an idea has been formed as the result of a process of reflection, and is capable of being applied equally to each member of a class of similar objects. When individuals are found to agree in the possession of certain common attributes which distinguish them from other individuals, we collect them into a class, and assign to it a name. The name serves not only to denote each individual of a class, but also to suggest the abstract notion of the circumstances in which these individuals were found to

agree. It is obvious that we can establish a scale of general notions, in which the names stand for relatively larger and larger classes of individuals with correspondingly fewer and fewer similarities. The terms employed to indicate higher and lower classes are genus and species, terms which are purely relative to each other. One of the chief features to be noticed in the nature of all ideas which have been formed in this way, is the fact that such ideas are constantly undergoing a process of modification in each individual's mind. The sciences of nature furnish numberless illustrations of this. Compare, for example, the scientist's idea of granite, or of a maple tree, with his idea of the same when a child. Compare your present idea of gravitation with your idea when you first heard of it. All ideas reached as the result of a process of observation and classification of facts are subject to this tendency. A single newly discovered fact may modify an accepted scientific theory. The discovery of a new species may necessitate a change in the connotation of the genus. It is not, however, by an imaginary separation of qualities of objects into like and unlike, and the formation of a class upon this

basis, that we reach the idea *six*. There is no group of qualities known under the name *six*, which is possessed equally by all the members of a class. When objects are numbered, *e.g.*, blocks, or books, or chairs, it is not in their character as possessing certain qualities, it is not as blocks, as books, or as chairs, that they are considered, but simply as discrete units, and hence as capable of being counted.* Further, we have no numerical ideas of which it can be said that they possess a higher degree of generality than others. The objects to which the term *six* may be applied are not restricted in any way by the presence or absence of certain qualities. And again, ideas of numerical relation differ from general notions in regard to the tendency to become modified. There is no such tendency to alter, there is no alteration possible in an idea of number, once correctly formed. After an analysis of what is involved in the number *six*, no new experience can add to, or in any way modify, our ideas concerning it.

6. Critics speak of the "style" of a literary production, meaning thereby an element which

* See "Comte, Mill, and Spencer," (p. 82) by Dr. Watson.

is due, not to anything in the nature of the subject itself, but to the way in which it was conceived and expressed, an element due to certain characteristics in the mental constitution of the writer. It may be safely affirmed that no two individuals conceive any given historical transaction in exactly the same way. If a statesman, a political economist, a military critic, a poet, a patriot, and an army contractor, were each to write an account of the battle of Waterloo, no two of the accounts would exhibit the same mental conception. Each account would be coloured by the personality of the writer. The facts would be regarded in different relations in each case. The points of view would be different. Such is not the case with arithmetic. The idea of any numerical relation, if apprehended rightly, is apprehended in the same way by all men.

When we say that numerical ideas are not generalizations from experience, that no added experience can produce any change in an idea of numerical relation once correctly formed, that numerical ideas, if apprehended at all, are apprehended in the same way by all men, that the

subject of arithmetic is not suited to the development of the power of imagination,—all this is expressed shortly by saying that arithmetic is an exact science.

7. A number is an abstract idea, and involves these factors: a *unit*, taken discretely, so many *times*, to measure a *collection or multitude* of things. The ratio between the unit and the multitude is the number. The expressions three, three pairs, three dozen, three score, all imply repetition. That which repeats or is repeated is different in each case. The unit or measure of the group, collection, or multiplicity of things to be numbered may be the simple unit, or it may be some other unit, a couple, a dozen, a score, or what not. In any and all of these cases, there is a relation discerned between the unit employed and the multitude of things to be measured. The relation, *e.g.*, between a group of things,—marbles, houses, or musical notes,—on the one hand, and the unit or measure on the other, may become the object of attention, and this relation is a number. We have the unmeasured multitude, the unit, and the (numerical) relation. It is clear that the same collection may be taken

in relation to any one of the possible units, and that with every change of unit we have a change in the relation, that is, in the number. The term ratio expresses the fundamental idea in numerical relation. A number is a ratio.

If the foregoing is correct; if the idea of number does not originate in experience by the activity of sense-perception; if it is not an idea compounded of different sensations: it must follow that number is a product of the thought power; and the laws which govern thought activity in general will be found to operate in our apprehension of numerical relation.

8. The thought process may for convenience be considered under two heads, analysis and synthesis, although these are in reality but two phases or aspects of the one mental process. All thought is at once differentiation, or the taking apart of what has been presented to the mind as a whole; and integration, or the combining of the elements thus distinguished. A complex is at first recognized as a plurality of constituents, and then certain relations are established among them. Let us enquire into the conditions under which the idea of number arises, and trace in

the natural history of the process of numbering, the operation of this analytic-synthetic movement of thought.

The condition under which the idea of number arises is the presence of a series of sensations. It is probably unnecessary to do more than refer, at this point, to the distinction between the origin of an idea of relation, and the occasion of its appearance in consciousness. Number does not originate in sensation. It originates in thought, in the act of establishing relations, but does not so originate except upon occasion of sensation. Attention is first given to the fact of repetition, and the first effort of analysis is directed to assigning to each sensation its place in the series. It is the idea of before and after, in which are held by one act of the mind the successive parts of a series. The relating idea, in other words, is that of *time*. Taking as a typical example the repetition of strokes upon a bell, analysis will not do more at first than concern itself with the task of relating one sensation as before and the next as after, in the succession. The two sensations are not present at once. Sensations are successive, not co-

existent. There is held before the mind, in the act which gives rise to the product, number, a unity of two members—the present sensation, and related to it, the idea of the one which has just passed away. So with sensations of colour and touch. Objects in space will occasion the activity of the mind in numbering, but it is to be observed that it is in the time relation that they are considered. The sensations, in whatever department of sense they originate, come before the mind in successive moments of time. It is only as discrete units, as successive items of experience, that spatial objects can be numbered, that is, related in time.

The effort of attention which does not at first extend beyond the mere idea of repetition soon results in isolating a portion of the series, the members of which are thought of as occupying the first and second places respectively, and then as a numbered group. Later experience occasions the incorporation of another member in the series, and the isolation of another group, the members of which are regarded as occupying the first, second, and third places, respectively. The significant feature to be noted in the process

is that it is only by an exercise of the mind's *relating* power that two or more of the series are thought of definitely as a group.

9. Formal logic recognizes the two aspects of thought known as conception and judgment, the former combining a group of constituents in a single whole, the latter holding out two elements distinct and yet related; and this view holds the concept as prior to the judgment. Judgment, however, is an original operation, and involved in every mental act. All conceptions include the idea of relation, which when explicitly set forth is a judgment. Judgment consists in making clear something that was obscure in the presentation. Again, reasoning consists in passing from a certain judgment, or certain judgments, to a new one, the distinctive feature of this process being known as inference. It is evident, then, that the essence of the whole process of thinking is in that form known as judgment. The elementary ideas of number can, as pure abstract ideas, be examined and analyzed so as to fully set forth all that is implied in them; the results of such analyses may be set forth as judgments; and chains of

reasoning may be formed from which new judgments will result.

10. The thought process could hardly be achieved without the aid of language. Whatever may be said of the possibility of retaining a clear idea of an individual thing as permanent, without the use of a word to stand for the thing, it is, at any rate, difficult to see how the apprehension of a class of things can take place without language. Certainly the retention of such idea, and the possibility of recalling it at convenience, depends on the name. It is usual to define a name as a mark or sign which may call up an idea in our own minds, or in the minds of others. It is not only a means of communication; it is also an assistance, perhaps indispensable, to the thought process itself.

A process of thought may be completed by attaching a name to the result. A similar process of thought may begin with the effort to think out the meaning of a term just now heard for the first time. What, exactly, takes place when we meet with a term which stands for an abstract idea, as *metal*, *constitutional government*, *triangle*, *six?*

In a very large proportion of cases, the name actually takes the place of the thought process. This function of the name has often been explained under the figure of a monetary transaction. Certain convenient tokens circulate in commerce, and are taken at their face value. There is an understanding among those who use these tokens as to what they stand for. Every one who uses them knows that he can, if he wish, exchange the paper for whatever amount of gold it represents. But this exchange is not made as an ordinary thing. Save in extraordinary cases, the transaction takes place without handling the gold. So, usually, with words. It saves time simply to recall the name, and this is easier than to reproduce in consciousness all the elements of the idea to which it corresponds. Along with the revival of the name, there is, probably, a more or less vague idea that, if we wish to take the time and trouble to do so, we are able to think out all that is meant by the name. The connotation of the class name "metal" does not rise into consciousness, in all its completeness, every time we use the word. Such complex ideas as are implied in the term "constitutional government"

are allowed to lie dormant, and the words themselves do duty for the ideas, in much the same way as do the symbols used in Algebra. Occasionally, analysis goes a little way. If the character of the discourse require it, we may stop and reflect upon the meaning of the term. We engage, under such circumstances, in recalling the actual thought process itself, as far as the occasion demands. The better acquainted we are with the true import of a term, the less need for analysis when we use it or meet with it. Frequently it happens that a term is met with, which we imagine ourselves to understand, and we are surprised to find as the narrative or conversation proceeds, that there is a hitherto unsuspected element in the idea, which it becomes necessary to account for. Reflection upon the exact meaning of the term shows that our supposed knowledge was incomplete. It is evident that the cure for imperfect knowledge is closer analysis. If a term be used which we have never met with, the process which goes on in realizing its meaning is one with that usually called conception.

The terms used to represent abstract ideas are thus seen to serve as substitutes for mental

processes, and this is true of all abstract ideas, mathematical ideas as well as generalizations from experience. In our ordinary use of mathematical expressions, as six, one hundred, one thousand, we recall nothing or almost nothing of the thought process. But we feel competent to think it out completely if we wish to do so.

11. If we enquire what is the real value of arithmetic, we shall discover the answer only when we have clearly realized the nature of number. If it is from thought, and not from sense, that we obtain a knowledge of things as numerable, if each phase of the thought power is called with activity in the formation of numerical concepts, if it is through logical control of thought processes only, that adequate knowledge of number can be attained, we cannot be wrong in concluding that this subject is suited to the culture of the thought power, and of no other.*

* " 'It would be proper, then, Glaucon, to lay down laws for this branch of science, and persuade those about to engage in the most important state-matters to apply themselves to computation, and study it, not in the common vulgar fashion, but with the view of arriving at the contemplation of the nature of numbers by the intellect itself,—not for the sake of buying and selling, as anxious merchants and retailers, but for war also, and that the soul may acquire a facility of turning itself from what is in course of generation to truth and real being.' 'A capital remark,' he replied. 'And, moreover, I now observe,'

Thought processes of whatever kind involve a special effort of attention. Such effort implies purpose, the attainment of some end viewed as desirable; and it implies also a certain degree of culture in the individual and in the race. Children find the operation of thinking about an abstract quality quite difficult compared with the mere sensible perception of a concrete object, or in the recalling of an absent one. Still more difficult is the process of thinking a relation between objects. In order to realize clearly the

said I, 'respecting that branch of science which concerns computation, how refined it is, and in many ways useful to us, as respects our wishes, if we will reply thereto for the sake of getting knowledge, and not with a view of traffic.' 'In what way?' enquired he. 'Just what we now said,—that it powerfully leads the soul upwards and compels it to reason on abstract numbers, without in any way allowing a person in his reasoning to advance numbers which are visible and tangible bodies; for perhaps you know of some persons skilled in these matters, who, if one were in argument to attempt dividing unity itself, would at once both ridicule him and not allow it; though were *you* to divide it into parts, they would multiply them, lest unity should somehow seem not to be unity, but numerous parts.' 'A very true remark,' he replied. 'But what think you then, Glaucon, if a person should ask them—you wonderful clever men, about what kind of numbers are you reasoning; in which unity such as you deem it, is equal, each whole to the whole, without any difference whatever, and having no parts in itself? What think you they would reply? This, as far as I think, that they speak of such numbers only as can be comprehended by the intellect alone, but in no other way. You see, then, my friend,' I observed, 'that our real need of this branch of science, is probably because it seems to compel the soul to use pure intelligence in the search after pure truth.'"

—*Plato's Republic, Book vii, Chap. 8.* (*Davis Tr.*)

nature of thinking, we may consider a few examples. A comparison may then be made between the various kinds, of which it is usual to distinguish those judgments which concern resemblance and difference, and from which result classification and definition of species; those which concern the relation of cause and effect; and those which relate to quantity and number.

The fusion of a number of concrete images into the so-called generic image, or type form including common features, is "largely passive,"* and probably accomplished without aid from language; for example, when the child has formed a mental image of a dog. Abstraction and comparison are involved in making the transition from this pictorial image to the conception properly so-called. When in the course of later experience of animals the child observes marked difference in size, colour, or temper, there will follow a series of judgments in which peculiar features already incorporated in the generic image are set aside, and a new and more definite idea formed. It is by processes of this kind

* See Ladd: Psychology, Descriptive and Explanatory, Chapter XIX.

repeated many times in the experience of the individual, and fixed in later stages by the help of language, that the true general notion is finally reached; and the term dog, metal, or animal, stands for this notion, which is found upon reflection to imply a certain definite set of qualities, and to apply to certain individuals.

After having noted a few repetitions of a certain sequence of events, a child will begin to attribute an observed effect to personal agency. If the incidents are of practical interest, and particularly if he himself is the actor, the element of expectation comes in to deepen the idea of relation thus established. Later on, cases occur which do not agree with his previous experience, and these stimulate and guide him in his transition from the earlier stage to one in which genuine inference is made.

It is probable that the establishment of numerical relations (see page 14) involves a greater idea of abstraction than judgments of resemblance and difference, those of causation or those of spatial qualities and relations; that the idea of a class of things, the question, "Who broke the cup?" the idea of a circle or a straight

line, all of them involve mental processes capable of being performed by a child who is unable to form the abstract idea of *six*.

It is true that young children may be taught to perform some interesting operations with marbles, spools, pebbles, etc., and, like the Japanese tradesman, with his *soroban*, get fairly correct results; but the operations are physical operations, and may be performed with equal speed and greater accuracy by a calculating machine. What is referred to here as involving a real effort of attention is the *mental* process of establishing a numerical relation.

The study of arithmetic, then, should be to the pupil an excellent means of mental discipline. The name of arithmetic has often been made a cover for many kinds of exercises from which thought is conspicuously absent. Its real function—that of giving employment to the faculty or power "properly denominated thought," though allowed in theory, has not generally been recognized in practice.

CHAPTER II.

RETROSPECT.

1. The use of the phrase, "the old methods," implies that there was an aim kept consciously in view by the school masters of a generation ago. This seems to have been to make pupils accurate and rapid in performing certain "operations," and in "solving problems." The "method" by which these results were reached may be described briefly. It consisted in plenty of practice in doing such work according to rule.

It would be unfair to say that this system prevented the possibility of exercising the reasoning power. A bright boy often discovered for himself the reasons which underlay certain of the rules, and "worked" the questions with a clear idea of the relation between process and result. There was nothing to prevent a kindly disposed or enterprising teacher from stepping out of the beaten course and making clear the reasonableness of a formula. It is enough to say, however, that while arithmetic was in the

stage of rules and formulas, a boy might go through the course successfully, might reach a high degree of skill in doing sums and problems, without much exercise of the reasoning power, *i.e.*, by a steady and slavish adherence to the prescribed rules. The tendency, with such aims professedly in view, was to discourage reasoning and encourage the mechanical performance of tasks consisting chiefly, if not wholly, in imitation.

Great emphasis was laid upon a "knowledge" of the multiplication table, as an indispensable tool in the performance of these tasks. The great thing was to get ready and keep ready for use one hundred and forty-four statements of abstract truth. These statements were committed to memory, parrot fashion. This procedure effectually prevented the exercise of the thought power in the consideration of pure number. There was no attempt to relate one truth to another. There could not be, for the simple reason that it was the words that were learned and not the truths underlying the words.

The mainstay of the general run of schools was not the teacher, but the text-book, and the

mainstay of the teacher of arithmetic was the key containing the answers to the problems. The chief defect in most text-books was that the subject was presented, not in the way in which it might best be apprehended by a young mind, but in the way in which a finished scholar might be supposed to review it. An example of this may be seen in the use of large numbers, long rows of figures representing something of which the child had, and could have only the faintest and vaguest notion. The only attempt made to clear up the meaning of these expressions was the early introduction of a chapter on notation and numeration. The result of this, however, was usually to increase, if possible, the confusion in the learner's mind between figures and numbers. When this took place, and there were not many cases in which it did not take place, there was a complete absence of thought proper on the part of the child engaged in the manipulation of long rows of figures. Figures and not numbers being the object of attention, it is easy to see that there could be no idea of relation, and hence none of numerical relation. The difficulty generally and erroneously supposed to reside in fractions was due to the same cause. The habit

of considering only the figures and ignoring the numbers, added to the evils of wrong definitions and illogical sequence in teaching, rendered that branch peculiarly dark and difficult.

In the working out of problems several errors are to be noted. The problems were based on matters foreign to the interest of the pupils (save only that attaching to the ever-recurring dollar or pound); the solutions consisted in the mechanical application of a rule or formula kept in mind ready to be applied to suitable cases, the cases being distinguished chiefly by the form in which the question was stated, and the student losing himself hopelessly if there were any variation therefrom; and the teacher usually overlooked the fact that the chief difficulty in many problems is to understand thoroughly the terms used in stating them.

The time at the disposal of the teacher was divided into two portions: one for mental arithmetic, the other for exercises with slate and pencil. Mental arithmetic consisted of the solution of problems involving smaller quantities than those given in the written exercises, but similar in principle with them, the memory

taking the place of the slate and pencil. Save for the greater demands on the attention due to the fact that imaginary figures are less vivid than those on the slate, "mental" arithmetic, when it was merely a reflection or shadow of the written, possessed about an equal value.

2. Obviously the chief defect in such a system lay in the confusion between figures and numbers. It was to remedy this that the system with which Grube's name is associated was devised. Instead of teaching numbers in general, notation and numeration, and the formal operations of addition, subtraction, multiplication, and division, with numbers larger than the pupils had any clear idea of, the new system proposed to deal with the smaller numbers first, give easy practice in the employment of all forms of calculation from the start, and use objects—blocks, marbles, etc.—as an aid to the formation of clear ideas of number and numerical operations.

In developing his system of teaching arithmetic, Grube seems to have adopted as his guiding principle the dictum of his predecessor, Pestalozzi, that "sense-perception is the basis of instruc-

tion." In this view of the case, (1) a number is like an object in space, and possesses certain qualities or properties which may be discovered by a careful exercise of the senses; it is only by means of the powers of observation that knowledge can be gained regarding the qualities of objects; and hence, in order to learn the properties of number, it is necessary to observe in the same way a number of blocks or other objects. (2) As number is something external to the mind, the arithmetical operations consist in the combination and separation of things, a mere physical putting together and taking apart. (3) The primary units composing any number possess spatial attributes, and the idea of fraction arises from the division of this primary spatial unit into parts.

(1) The principle enunciated by Pestalozzi,—sense-perception the basis of instruction,—underwent considerable extension at the hands of Grube. When applied within its appropriate sphere, the principle is most useful. One of Pestalozzi's aims in education was "to popularize science," and in that sphere the principle has a very obvious application. It is indeed

difficult to imagine how any progress in science could be made, if scientists were not trained to observe the facts of nature. It is easy to see that a student of any one of the sciences of nature will work at a disadvantage if his sense organs happen to be weak or untrained. A high degree of intellectual as well as perceptive power is necessary here; but other things being equal, the scientist whose sense organs are in perfect condition has an immense advantage over one not so endowed. That an object is of a certain colour, gives forth a certain sound when struck, possesses a certain degree of hardness or roughness, has a sweet, bitter, or salt taste, gives off a pleasant or unpleasant smell,—these are facts to be observed by the scientist, and there is no way at present known to man by which such facts can be observed save by the exercise of the perceptive powers. There can be no doubt that in the domain of the sciences of nature, sense-perception is the basis of instruction.

In adopting this as a guiding principle and unwarrantably extending its application to the science of number, Grube seems to have had before his mind two circumstances. In the first

place he noticed that one of the chief faults in the teaching of science—the memorizing of mere words—was also a gross defect in the teaching of elementary arithmetic. In the second place he saw the unwisdom of insisting upon the pupil's mastering the operations of addition and subtraction before going on to multiplication and division; and casting about for an illustration to show the defect of this procedure, he thought he discerned a parallel in the way in which we acquire a knowledge of natural objects. It is not by noticing the same quality in different objects, but by observing one object thoroughly that a child comes to a clear intuition of the plant or other natural object. Misled by the fact of a common defect in teaching, as well as by his own illustration, he came to the conclusion that the dictum of Pestalozzi applied to arithmetic. He accordingly decided to study a number as he would study a natural object. For him, a number is a thing in pretty much the same sense that a plant, or a stone, or a shell is a thing. A plant occupies a certain portion of space, and may be seen, touched or tasted. A number is a group of things and occupies a certain portion of space. It may be examined,

not in exactly the same way as a plant but in a very similar way. The more thorough your observation of the plant, the more valuable will be your knowledge regarding it. The consideration of a number ought to be thorough for the same reason.

Grube would not go so far as to say that any department of sense perception can give us an idea of number in the same way that the eye gives us ideas of colour, but he makes an equally fatal error. Having, as he thought, established a parallel between the genesis of our knowledge of a number and that of our knowledge of an object, he becomes blind to everything in the process save the part played by sense perception. He recognizes that without a series of sensations we can have no idea of number. He forgets that the idea of number could not arise but for the work of the mind in introducing order and relation into what is given to us by the senses. He must have known that relations are not sensations; but he failed to provide for that very exercise of the thought power without which an idea of numerical relation can never rise into consciousness. His analysis of the number idea

goes far enough to find out that its essential "property" consists in the "way" in which a group of things may be put together and taken apart*; but it does not go far enough to show that this essential property of number is the element which is contributed by the thought power, and is not in any manner due to the senses. Number is a "way" of thinking about objects, not an inherent property which may be discovered by manipulating them.

(2) A number, then, is not a thing, or a group of things. The process known as numbering is not a physical process; it is a mental one. A pupil who can think of five as made up of two and three, may illustrate this truth by holding up his five fingers. On the other hand, he may learn to perform the operation of putting two objects with three objects without the mental process which the operation is supposed to embody. Grube, however, having decided to treat a number as a mere aggregation of spatial

*J. S. Mill takes a similar view :—"What is then connoted by the name of a number? Of course, some property belonging to the agglomeration of things which we call by the name; and that property is the characteristic manner in which the agglomeration is made up of and may be separated into parts."

--*Logic: Book iii, Chapter 24, § 5.*

units, as a mere something external to the mind and its processes, was naturally led to make physical operations do duty for psychical ones. Addition, the first arithmetical operation in his scheme, appears to consist in putting the two objects with the three objects. Numbers being thought of as simply groups of objects, the piling up or separating of such groups is supposed to be, not merely to represent, addition and subtraction.

Grube has signally failed to understand that a number is a relation imposed upon things by the mind's action; that the symbol *six* expresses, not an object or group of objects, but an idea. An idea is a mental process. When we are attending to any idea of *relation*, we have before us two constituents connected with each other in a certain way. That which is the especial object of thought is the connection between the constituents, not the constituents themselves. What are the constituents involved in an idea of number? There is the multitude, on the one hand, and a unit on the other. That which occupies the attention is the connection between the multitude and the unit.

Reflection upon what is involved in any idea of number will give prominence to one phase or aspect, while the other phases or aspects retire to the background. The unit by which the whole is measured may be the simple unit, or it may be a complex unit. The kind of relation which is established or discerned between the two constituents of the idea may be the simple relation in which ratio is implied, or it may be that in which the ratio is explicit. As the establishment or the discernment of the relation involves a judgment, this may be set forth in a proposition. The idea expressed by the term *six* may be taken to illustrate this. The relation between the whole quantity and the simple unit is one phase of the idea. The proposition, $6-5=1$, expresses one kind of relation; $\frac{1}{6}$ of $6=1$, expresses the other kind. What is wanted is logical control of these thought processes.

All propositions set down to express relations between quantities and units may be classified as representing some one of the fundamental operations of arithmetic. There are operations which consist in piling heaps of objects together, and separating them into smaller heaps. Such

operations are to be distinguished from the psychical operations which have just been described.

(3) The idea of a fraction as a portion of a primary spatial unit is quite consistent with that mistaken idea of a number which governs the theory of Grube.

He would define a fraction as "a part of anything." The idea of fraction, in this view, arises in the mind when any object occupying space, and taken as a unit, is divided into an equal number of parts. Thus of a number of apples we may take one and divide it into three parts. One of these parts is properly known by the name of one-third, and two of the parts, two-thirds.

It should be tolerably clear at this stage of our discussion that such a view of the nature of the fraction is erroneous. The errors involved are precisely those which have already been pointed out in the idea of number as conceived by the exponents of this system. The fundamental defect is the assumption that there is a space element in number.

Number is the repetition of a unit, *i.e.*, of a

measure. The most careful search fails to reveal any space element in such a concept. A recent work on number states that measurement with "an undefined unit," as the unit *pace*, for example, is a "stage behind" that which employs the unit *yard*, because the yards are all equal to one another and the paces are not. "When we pass to measurement with exact units of measure, this idea of fractions—of parts making up a whole—becomes more clearly the object of attention. The conception, 3 apples out of 5 apples has not the same degree of clearness and exactness as that of 3 inches out of a measured whole of 5 inches. Why? Because in the former case we do not know the exact *value*, the *how much* of the measuring unit; in the latter case the unit is exactly defined in the terms of other unities larger or smaller; in 3 apples the units are alike; in 3 inches the units are equal.*

The error here is in supposing that the spatial equality of the actual things which are taken as units is brought into consciousness at all. As apples, or as inches, the spatial aspect may

*McLellan and Dewey, Psychology of Number, p. 128.

receive attention; as units, every other aspect is set aside, and that aspect alone is considered which they possess as units—an aspect which they would not possess but for the action of the mind in so constituting them. For various reasons it has been found convenient to agree upon certain units, as inches, ounces, etc. Very great efforts are made to secure uniformity in their use, and a high degree of exactness is reached. For all practical purposes these inches, etc., may be regarded as equal. They approach, of course, very much nearer to absolute equality than apples or paces. They are as nearly equal as human devices can make them; but as absolute equality cannot be reached, it is necessary to take them as equal. In practical measurement there is a difference in degree of exactness between the pace and the yard; in numbering, the pace receives its value as unit, from the mind, in precisely the same way as the yard, or other "measured quantity," receives its value. The wisdom of using the convenient yard rather than the pace, in actual measurement, is not called in question. The position taken is simply that the conception—3 apples out of 5 apples—does possess the same degree of clearness and

definiteness as the conception, 3 inches out of a measured whole of 5 inches. The mental act which gives rise to the numerical product is the same in the cases given. The units receive their constitution as units from the mind; they are placed in precisely the same relative position; they possess one and the same value; they are thought of as identical. Unless taken "as" equal, the paces are not units at all.

A unit is properly defined as a measure. The term simple, applied to the primary unit, emphasizes the fact that it is indivisible, ultimate, simple. How does this unit come by its peculiar character of simplicity? Granted that the strokes on a bell may be numbered by relating each sensation in its place in the series, and that the units so constituted possess this simple individual character acquired by the apple (or the inch) used as a unit, how does it lose its spatial attributes? The answer is, that in numbering objects in space the mind considers them in the time relation, as a series of discrete units, and must do so—owing to its own constitution —before they can be numerically related.

The idea of fraction, then, cannot possibly

arise from the division of the primary unit, the distinctive feature of the primary unit being that of indivisibility. In this respect it recalls the atom of the scientist, which is ultimate, and the point of the mathematician, which has no parts and no magnitude. The numerical unit is not an object in space whose spatial attributes are to be kept clearly in mind in the act of numbering, and from which the idea of fraction arises in the process of dividing it into parts. All fractions are numbers, and the idea of fraction arises in the same way as that of any other number. The idea of fraction is the idea of the relation of a unit to a multitude. As has already been pointed out, the complete idea of number, when all that it implies is fully set forth includes that of ratio. The ratio becomes prominent when the fractional relation is discerned and expressed; and it may be expressed either as $\frac{1}{6}$ or as 1 : 6.

3. It has been pointed out that the chief defect in the teaching of arithmetic prior to the advent of Grube lay in the confusion between figures and numbers. It is not necessary to do more than refer to this fact. No

one to-day attempts any serious defence of the old system. To attack it is to waste energy upon an abandoned position: The recoil, however, from the system of figures, though eminently desirable, was too violent. As often happens, undue emphasis was placed upon the other extreme. If we have hitherto neglected to take into account the part played by sense perception in the production of the idea of number, said the reformers, let us by all means proceed to remedy that defect. If the idea of number cannot arise in the mind without the intervention of objects, let us see to it that objects are provided. Accordingly, the handling of objects—blocks, marbles, stones, etc., was encouraged, but to such an extent, unfortunately, in many instances, as to obscure the psychical process itself. The really important consideration is the psychical process. At best the physical process of putting two blocks with three blocks could serve no purpose beyond that of illustrating the psychical one. Somehow or other it had got to be assumed that number was in the objects, and could be, and had to be, extricated from them by the mind, and that this was to be accomplished only by placing the

objects in certain positions relative to each other. The error was in "assuming there are exact numerical ideas in the mind as the result of a number of things before the senses"*; and this assumption was due to an incorrect theory as to the origin of the idea of number, namely, the assumption that it is due wholly to experience.

In the fulness of time, it was pointed out that the measuring idea in number had been neglected. By this measuring idea in number is meant that the function of the unit is to measure, that in numbering any collection of things, by any unit, the exact number is found when the whole group has been defined as so many of the unit, that in short the unit measures the collection or group so many times. There are a thousand men in ten companies of a hundred each. When the group is thought of as one thousand our unit is the simple unit; when thought of as so many companies, our unit is one hundred times what it was before. In thus numbering we have *measured* the group, though with different units in each case.

*Psychology of Number, McLellan and Dewey, p. 62.

When the importance of this feature — the measuring function — was recognized, efforts were at once put forth to secure the psychical activity from which the idea of number, so conceived, might be expected to arise.

"If, to help the mental process, small cubical blocks are used to build a large cube with, there is necessarily continual and close observation of the various things in their quantitative aspect; if splints are used to enclose a surface with, the particular splints must be noted. Indeed this observation is likely to be closer and more accurate than that in which the mere observation is an end in itself. In the latter case there is no interest, no purpose, and attention is labored and wandering; there is no aim to guide and direct the observation." *

It was necessary that the "method of symbols" should be abandoned for something better. Experience has proved the results to be as barren as might reasonably be expected from a method so artificial and mechanical. It was necessary, too, that the measuring idea in number should receive due emphasis. But the course

* McLellan and Dewey, Psychology of Number, p. 62.

proposed in the latter case, though right in purpose, was as wrong in method as that in the former. The handling of objects merely provided certain exercise for the senses, and the examination, comparison, and measurement of extensive magnitudes simply gave another direction to this sense activity.

The building of a large cube with smaller ones is an interesting occupation. It requires "continual and close observation." Eye and hand alike are trained in the process. A very high degree of accuracy may shortly be reached by the observer engaged in the occupation of estimating what are technically called "the attributes of sensation"—that of "extent" among the rest. Exercises in measuring either with the eye or with a yard stick, as they demand close observation, tend unmistakably to the development of finer sense discrimination. The more interested the observer becomes the more chance for development of the kind mentioned. Unfortunately, however, the more completely the child becomes immersed in the process of measurement and comparison, the farther he is carried away from the idea of number. Just to

whatever degree the attempt to secure interest in the comparison of extensive magnitudes is successful, to that degree is the exercise destructive of clear ideas of numerical relation. Delicacy of sense-discrimination is a desirable thing, but, as we have already pointed out, it is in no way connected with clear ideas of number. Here, again, it is only necessary to refer to the fact that any theory which assumes the existence of a space element in number will inevitably involve erroneous conclusions regarding the exercises which are necessary or suitable to the development of the idea. It has been shown that analysis fails to discover any such element. Whatever useful ends may be served by engaging in exercises of this kind—the examination and comparison of space forms—numerical accuracy is not one of them.

CHAPTER III.

THEORY.

1. In all psychical activity there is a two-fold movement, which it may be profitable to trace through the various stages of intellectual progress.

In the case of sensation we observe that there is implied, in addition to the peripheral excitation, a certain responsive action necessary to the working up of the raw material into well-defined psychical states, and this central responsive action is called attention. We are said to attend when certain of our sensations, or psychical phenomena of any kind, are brought forward into clearer relief, while others are allowed to withdraw into the background of consciousness. It is an activity which has been well described as primarily a process of adjustment, of concentration, or narrowing, of the psychical area. It has a second function as well: that of combining a plurality of psychical elements. We may thus concentrate our minds

upon one of the constituents in a chord, or attend to the complex effect as a whole.

Again, we find that the two functions just described—distinguishing sensations from each other, and combining them into a group—have a parallel in the phases or aspects of primary intellection. Primary intellection involves differentiation or the singling out of the constituents of any complex; and integration, or the conjoining into a whole, of the elements so distinguished. In the primary stage of the organization of our experience—the formation of the percept and its ideal representation, the image—intellectual development proceeds along this double track of separation and combination.

Further, there is to be found in the formation of thought products, what is simply a higher development of the same double process. Thought is one operation, having two sides or aspects: Analysis, the taking apart of what enters into a complex, and Synthesis, the putting together of the elements to form a related whole. In this highest phase of mental activity, the two-fold movement which we have been considering, though the same in nature with

what has been described in the more elementary phases, is found to require a much greater effort of attention. The isolation of some particular feature in a presentation, when this feature is prominent, does not demand any special effort of attention. When, however, there are prominent features calling for notice, and the effort is to attend to some minor one which does not strongly assert its presence, we have abstraction, or abstract attention. And this is precisely what takes place in the thought process. It involves a high degree of abstraction, and results in the formation of those highly elaborated products known as concepts and judgments. As has already been pointed out, the essence of the whole process of thinking is in that form known as judgment, and an examination of any judgment will show that the process consists in *distinguishing* two factors in order to *relate* them to each other.

It is not considered necessary to enter into an elaborate defence of the position that analysis is the primary and preparatory, and synthesis the complementary stage of all mental processes. The priority of the analytic phase in all psychi-

cal activity is assumed. No thinker of any repute has ever advanced or tried to maintain the contrary opinion.* This well-known law—

*"Analysis and synthesis, though commonly treated as two different methods, are, if properly understood, only the two necessary parts of the same method. Each is the relative and correlative of the other. Analysis, without a subsequent synthesis is incomplete; it is a mean cut off from its end. Synthesis without a previous analysis is baseless; for synthesis receives from analysis the elements which it recomposes. And, as synthesis supposes analysis as the requisite of its possibility, so it is also dependent on analysis for the qualities of its existence. The value of every synthesis depends upon the value of the foregoing analysis. If the precedent analysis afford false elements, the subsequent synthesis of these elements will necessarily afford a false result. . . . .

"The two procedures are thus equally necessary to each other. On the one hand, analysis without synthesis affords only a commenced, only an incomplete knowledge. On the other hand, synthesis without analysis is a false knowledge—that is, no knowledge at all. Both, therefore, are absolutely necessary to philosophy, and both are, in philosophy, as much parts of the same method, as in the animal body, inspiration and expiration are parts of the same vital function. But though these operations are each requisite to the other, yet were we to distinguish and compare what ought only to be considered as conjoined, it is to analysis that the preference must be accorded. An analysis is always valuable; for though now without a synthesis, this synthesis may at any time be added; whereas a synthesis without a previous analysis, is radically and *ab initio* null."

—*Sir William Hamilton: Metaphysics, Lecture V.*

"If a part be conceived without any reference to the whole, it becomes itself a whole—an independent unit; and its relations to existence in general are misapprehended. Further, the size of the part as compared with the size of the whole must be misapprehended unless the whole is not only recognized as including it, but is figured in its total extent. . . . Still more when part and whole, instead of being statically related only, are dynamically related, must there be a general understanding of the whole before the part can be understood. By a savage who has never before seen a vehicle, no idea can

all thought begins in analysis—governing as it does the whole range of intellectual activity, is surely one of first-rate importance. Pedagogical science attempts to deduce from the facts of mental life a system of rules for the guidance of the teacher. This is perhaps the most signifi-

be formed of the use and action of a wheel. . . . Most of all, however, where the whole is organic, does the complete comprehension of a part imply extensive comprehension of the whole. Suppose a being ignorant of the human body to find a detached arm. If not conceived by him as a supposed whole instead of being conceived as a part, still its relation to the other part, and its structure would be wholly inexplicable. . . . A theory of the structure of the arm implies a theory of the structure of the body at large. And this truth holds not of material aggregates only, but of immaterial aggregates—aggregated notions, thoughts, deeds, words."

—*Herbert Spencer: Data of Ethics, Chap. 1.*

"Successive states of consciousness may be represented as waves, of which one is forever taking the place of another, but such successive states cannot make a knowledge even of the most elementary sort. Knowledge is of related facts, and it is essential to every act of knowledge that the related facts should be present in consciousness. . . . A known object is a related whole, of which, as of every such whole, the members are necessarily present together."

—*Green: Prol. to Ethics, Book I, Chap. 2.*

"Just as we saw that all intellectual elaboration is at once differentiation or separation, and integration or combination of what is differentiated, so we shall find that thought itself is but a higher development of each phase. First of all, then, thought may be viewed as a carrying further and to higher forms the process of differentiation of presentative elements by means of isolating acts of attention. . . . In the second place, all thought is integrating or combining; or as it is commonly expressed, it is a process of Synthesis. In thinking, we never merely isolate or abstract. We analytically resolve the presentation complexes of our concrete experiences only in order to establish certain relations among them."

—*Sully: The Human Mind, Chap. 9, § 3.*

cant psychological law with which the teacher has to reckon. For him it becomes a rule or guiding principle. The entire realm of pedagogical science may be ransacked in vain to discover a principle of equal value. It is the golden rule of teaching.

Having shown that this two-fold movement belongs to every intellectual act from the highest to the lowest, it will now be necessary to consider it in relation to mental progress and development.

2. The history of the mental experience of any individual, when viewed as a whole, exhibits an adjustment of means to an end, a stream of consciousness, self-directed in part, and in part impelled, to the accomplishment of a certain work. The flow of this stream is an unbroken one; every wave is what it is by reason of its relation to every other; and the currents and eddies tend more and more to become persistent. Continuity, relation, and tendency to solidarity are the obvious features of the intellectual life in its development.* Written out at length, the history of that development would be that of a

* This is well worked out by Ladd, in his "Psychology, Descriptive and Explanatory," Chapter XXVII.

continuous series of analytic-synthetic acts, related to each other in a definite way, the whole course marked by the gradual establishment of habit.

The law of habit is exemplified by every act of every human being in course of development, whether at work or at play, standing or walking, feeling, desiring, perceiving, or thinking. As affected by the operation of this law, the analytic-synthetic principle seems to take on additional importance. A habit of thought marked by incorrect or inadequate analysis means simply the acquisition of a stock of vague and ill-arranged concepts; whereas habitual logical control of such processes will ensure clearness and orderly arrangement of one's intellectual stores.

Turning now to the two features of continuity and relation in mental development, we may include all that is implied in these principles in the statement that the whole of mental development consists in an unbroken series of interdependent psychical processes. The point of interest for us is to discover the exact nature of the relation referred to. What exactly is the

relation existing between one wave of progress and another, in the stream of conscious life? Is it possible to formulate a statement regarding the conditions of mental progress which shall sufficiently set forth the important features of the process, and at the same time avoid that vagueness which is a frequent accompaniment of general statements?

We must keep in mind the fact that the series of psychical acts which go to make up the mental life of an individual is a series of acts each of which includes a two-fold movement; and also that in this two-fold movement analysis is the prior, and synthesis the complementary phase. We are to remember that an analytic process implies the presence to the mind of some complex, some whole, awaiting this process. Consciousness is always the consciousness of a complex. Our picture of mental life then would be that of a series of wholes, each of which undergoes a more or less complete analysis, and a subsequent synthesis. Further, as we are constantly making additions to our mental stock, and assimilating and incorporating new and old together to form new concepts, we recognize that

all new ideas thus formed will receive their character and complexion very largely from the old, that adequate knowledge depends quite as much upon the character of what we already possess as upon what we now acquire. A total mental product, made up of a number of factors, will necessarily partake of the character of those factors. If the factors are clearly understood in the first place, and their relation to the new concept clearly apprehended, our new knowledge will be clear and distinct. If not, we must content ourselves with inadequate knowledge, or else employ the natural remedy for such, *i.e.*, closer analysis.

3. We are now in a position to gather up the results of this enquiry, and set forth the conditions of mental progress as follows :—

All thought begins in analysis and ends in synthesis. This is true of the mental operations of the youngest child. It is equally true of the mental operations of the oldest and wisest philosopher. The progress of human knowledge, then, may be considered as a series of points at which the mind considers wholes; and its power to understand these wholes depends upon the

extent to which it has previously mastered, as wholes, the parts which now make up this more complex whole.

A student is trying to understand the theory of winds. To compass this task he must perform a certain analysis. What is involved in that theory? His power to understand the complex problem now presented to him depends precisely upon the extent of his knowledge of gravitation, and the expansive force of heat. No difficulty is experienced where these factors have been previously grasped. But an acquaintance with them is indispensable to any one who desires to understand the theory of winds. At some previous stage in his career the student may have heard of both of these matters. But his ideas concerning them are indistinct. It may be, for example, that he has not fully realized the fact that the force of gravitation and that of heat are exerted upon air as well as upon solid bodies. Before he can make any progress with the theory of winds he must first devote his attention to these factors, in turn dealing with each as a separate whole, and analyzing each into its separate constituents. Then, and not till then,

can he approach the greater whole with any prospect of comprehending it.

A student of British history unacquainted with the fundamental laws of the realm, as vindicated by the barons at Runnymede in the thirteenth century, and by the Long Parliament in the seventeenth, can form no clear idea of the significance of the Revolution of 1688. Why? Simply because as already set forth, knowledge of any whole necessarily includes knowledge of the parts which go to make up that whole. An historical transaction can be comprehended in its entirety only when the factors have been previously comprehended. The disputes regarding the dispensing power, standing armies, the right of petition, arbitrary imprisonment, imposition of taxes, freedom of elections, possess no meaning to the student of history except in the light of the great charters.

Examination of the sentence in grammar shows that there has been an act of judgment set forth in words, that two ideas have been distinguished and related. No effort to understand the expression of a judgment can be successful without a previous clear understand-

ing of the concept as expressed by a term. The question whether a given expression should be classed as a sentence or no cannot be answered by one who is unacquainted with the elements of the judgment.

The position taken is that the examples given are illustrations of the application of a principle of universal scope. The application of this principle is not circumscribed, is not confined to any one science or group of sciences. It obtains in every department of human thought. The master mechanic in a great workshop who has been advanced from post to post, from simpler to more complex duties, occupies his present position by reason of the fact that he has mastered, as wholes, certain elementary tasks belonging to the previous stages of his experience; and his fitness to perform his larger tasks rests fundamentally upon the use he is now able to make of adequate knowledge previously acquired. A man who has failed in carrying on his business is usually an example of failure to pay due heed to the rule implied in this principle. Ignorance or inadequate knowledge of affairs of detail means inability

to sufficiently understand matters of larger concern.

If the principle under discussion demands recognition for the prosecution of studies in the inexact sciences, if close adherence to the rule it suggests is necessary in the departments of thought from which illustrations have been taken, what shall we say of the importance of its application to an exact science like arithmetic?

4. It is necessary at this stage in our discussion to notice some important features of educational theory regarding the arithmetical operations, addition, subtraction, etc.

Thought processes involving numerical calculation conform to the general type commonly named analytic-synthetic. Each case of an arithmetical operation will be found on examination to involve both the analytic and the synthetic phase of thought. The proposition "eight is seven and one" expresses this twofold movement. Analysis of the concept "eight" results in the recognition and distinction of the two factors. Eight is thought of as separable into seven and one. The complement of the

idea that eight is so separable is that these parts together make the whole; and the thought is not complete until both aspects have been considered. Again, the proposition, "eight is four twos," obviously expresses the same two-fold movement, the initial idea being the separation into factors, the concluding one that of combination to form the whole. The proposition "one fourth of eight is two" similarly involves these features.

5. The arithmetical operations are analytic-synthetic operations, and of these the terms subtraction and division seem to apply to and emphasize the prior, or analytic, while addition and multiplication refer to the latter or synthetic phase of the process.

It is often assumed and sometimes stated that in the acquisition of ideas of numerical relation, the mind proceeds by the addition of another unit to the last learned number; that one who knows the number "two" gets to know the number "three" by adding "two" and "one" together, or that the idea "two" is reached by putting "one" and "one" together. If the idea of "two" is really reached in this way, that is to

say, if a child gets to the idea of "two" by thinking of one and afterwards of one more to be added to the first one, then we have discovered an important exception to the operation of the supposed universal law which we have been considering, namely, that in the two-fold movement of mind analysis is the prior and synthesis the complementary stage.

The procedure of the mind from "two" to "three," or from "three" to "four," is sometimes spoken of as a synthetic process. Let us consider how the mind really acts in taking this step. During the time that "two" is being considered, it is not thought of in relation to any higher group. When the group "two" has been grasped, the mind is ready to deal with the more complex problem which some external agency presents to it.

The proposition, "two is made up of one and one," is the expression of a judgment. It has already been shown that judgment is an original and primary act of mind, that it consists in the distinction and relation of two factors, and that in every such act the elements are held before the mind as *at once* distinct and related. The

elements spoken of may be parts of a complex of colour, form, etc., *i.e.*, a datum of sense; or they may be elements of an abstract idea. In either case the purpose and effect of an act of judgment is to make clear something which was obscure in the first place. Are we entitled to hold that the proposition "two is one and one" is the expression of just such a judgment as has been described? The position taken is that we are entitled to do so, and that in reality there is no exception to the analytic-synthetic principle. What is present to the mind first is the idea of "two." It is by a process of analysis that the constituent elements are made clear. We never find ourselves in possession of a perfectly simple mental experience: consciousness is always the consciousness of a complex.

Any theory of number, then, which takes addition to be the first in logical or psychological order of the numerical "operations" must deny the priority of the analytic phase of thinking.

It is true, certainly, that no ingenuity can succeed in inducing the mind to violate its own laws. This law of mind is as inexorable as gravitation. But it is also true that the mental

energy of the learner may be wasted through the effort of his teacher to induce him to violate it. Materials for thought may be presented in such a way as to encourage analysis and discourage synthesis. The emphasis which has been laid upon the priority of analysis should not obscure the fact that synthesis is its necessary complement. The mind in all stages of its progress disintegrates its materials for thought in order to establish relations among them. The first movement seems to stand to the second in the relation of means to end. The intimate and necessary connection between them shows the absurdity of employing, or attempting to employ, or emphasizing one phase for a considerable time to the exclusion of the other.

It seems to be taken for granted by many that there is a great difference in point of difficulty between the operations of subtraction and addition, and the corresponding operations of division and multiplication. It has even been stated that the child has no actual experiences which suggest to him the necessity of the division operation, and we are solemnly warned of the danger of "forcing these operations into

consciousness." * As if a child who had been told by his mother to carry in two pails of water, and being unable to carry more than one at a time, would not, if asked, be able to state how many times he would have to bring in one pail of water.

It has been noticed that students reach the High school quite unable to explain the difference between addition and division. It will be found that the theory of a space element in number, and an undue emphasis laid on synthesis, will sufficiently explain the confusion which has grown up in the minds of such students.

That analysis and synthesis are of the very essence of all intellection, that analysis is the first and synthesis the succeeding stage, and that this is the order in the case of every single thought process,—there is a greater degree of unanimity in the acceptance of these statements as theoretical truths, and in their violation as practical principles than in anything else of the kind with which we are acquainted. The study of number carried on in accordance with these

* McLellan and Dewey, Psychology of Number, p. 100.

principles consists in the solution of a series of problems which present themselves naturally to the learner. The process which he will be called upon to perform will not be a set of exercises chosen arbitrarily and without reference to the demands of the mind or of the subject matter. There are problems in connection with the study of number which the child will, if properly led, if his thoughts are not misdirected, learn to discover for himself.

6. Not infrequently there are to be found individuals who, desiring to be considered practical in the highest degree, put forth some such theory as the following: The elementary truths of number are not very numerous. They may be systematically collected in the form of addition, subtraction and multiplication tables, which for all practical purposes will include the truths of number. The pupil should be made to learn these truths off by heart, and learn them thoroughly. It is necessary that they should be learned in such a way as to be instantly available. It will require considerable drill and constant review. Thus by a judicious use of

the power of memory the desired result may in a very short time be attained.

This proposal starts with the assumption that arithmetic is to be studied with an eye single to its so-called practical value, as enabling the student to get answers to questions that arise in daily life speedily and correctly; and it involves, moreover, a very wide-spread and pernicious error regarding the nature of the process called "review."

It has been frequently remarked that arithmetic is a thought study. What is the character of the knowledge which the pupil possesses where that knowledge is confined to "tables" learned in the usual way? What is involved in the ready employment of those tables? The knowledge is simply a knowledge of words. A call has been made upon artificial memory. The pupil's entire numerical apparatus is built up of words. If he have a good memory for words, he will, to the casual observer, appear to possess real knowledge. But it is merely the semblance of knowledge after all. It is not a knowledge of number, but a memory of words. The thought power is not in the very smallest degree

called into play. To revert to the figure of language as currency, the language of tables so learned is counterfeit coin. It pretends to be what it is not. It has the appearance, but not the reality, of numerical knowledge.

Review does not consist in merely beating a path already traversed. It is rather the process involved in viewing a bit of country first from one standpoint and then from another, ascending to greater and greater altitudes in order to take larger and more comprehensive surveys, each of which will include all the preceding ones. The point to be noticed is that we are thus constantly seeing old things in new relations. In order that progress may be made, old truths must constantly be employed in the discovery of new. To review, then, simply consists in making this use of truths already learned. The proper review of any subject is that review which is necessarily involved in the procedure marked out by the dictum (p. 55). The acquisition of a new idea has been shown to depend upon our previous mastery of the elements which compose it. To grasp a new idea, then, we are forced to consider it as a whole made up of these

elements. No idea can be acquired without this analysis. And what does this analysis involve but a reconsideration of previously learned factors? This is the true method of acquiring new knowledge, and it includes the true method of reviewing the old. A review of arithmetic which consists in the mere repetition of thought processes without any effort to incorporate new truths with the old is not a true review. To march back and forth in this way is about the same thing as marking time.

CHAPTER IV.

PRACTICAL.

1. We have now to consider a matter of the largest practical importance—the order of steps to be followed in the scientific study of number.

In what order shall the numbers be considered, and to what extent shall each number be studied? A moment's consideration of what is implied in the principle laid down on p. 55 should make it clear that there is no course open to us but a consideration of the numbers in the order of their magnitude, and an exhaustive study of each number in turn.

The first of these points requires no further remark. The second will need some elucidation. What is here meant by an exhaustive study of a number? Briefly, such a study as will meet the conditions of clear thinking. These have been set forth in the form of a general principle. It will be necessary, however, to show somewhat in detail what is meant by clear and adequate knowledge of a number.

The relation between the concept and the judgment has been discussed sufficiently for the purpose in hand. Some important considerations regarding the functions of language have also been noticed. It has been pointed out that terms employed to represent mathematical (as well as other) ideas serve also as substitutes for thinking. Accompanying the use of any mathematical term is the vague consciousness that we are quite able to think out and state if we wish to do so, all that is involved in the thought process by which the idea was reached. Without the actual process of judgment, the so-called numerical concept can have no existence in the mind, and apart from the ability to recall this process of judgment the name is a mere counterfeit coin. In order to be ready for all possible emergencies, one should be able to recall every one of the series of judgments involved in the concept.

Any given group may be thought into its factors. Several judgments of this kind may be performed in the case of the smaller numbers. Thus six may be thought of as five and one, as four and two, as three and three. In the

case of larger numbers, a great number of such judgments may be made. What is lacking in a numerical concept which has been built up exclusively of judgments having reference to the parts into which a group may be divided? It is incomplete in that attention has not been sufficiently directed to the number as a whole. It is necessary to consider six as made up of four and two, of three twos, of two threes, and of six ones. Here, the emphasis is upon the units which measure the whole. It may be added that a clear idea of number requires complete analysis of the unit employed: and this is true of all numerical ideas whatever. A clear idea of ratio—which is of the very essence of number—requires that the idea of the whole shall be fully grasped. Not only must we consider the group six as made up of six ones, or as measured by one six times. We must now emphasize the fact of the equality of these units, and so reach the judgment "One is one-sixth of six."

Adequate knowledge of a number includes, then, those judgments which are reached as a result of the measurement of a group by all its

contained units; and such judgments will fall into two classes: those in which what may be called integral relations only are involved, and those in which the idea of ratio is explicitly set forth.

2. An illustration or two will serve to show the practical application of what has been laid down. The pupil is beginning the study of *three.* He has already mastered *two.* What is the problem now presented to his mind? The problem is the problem of measuring the new whole by its contained units. It has been pointed out that these measurements will result in the formation of two classes of judgments: judgments involving integral relations, and those involving fractional relations; answers to the questions, how many? and how much? When this is accomplished the learner will come nearer to perfect knowledge than is possible in the case of anything else we are acquainted with in the whole range of human thought.

Following the line of least resistance, what is the simplest measurement of *three* that can be made? Will the pupil naturally measure the group by *ones,* or will he employ as a unit the

group *two* which he has just analyzed? The answer to the question as to which of these processes is the more complex will guide us into the path of least resistance.

In the proposition, "Three equals two and one," there is involved the idea of the whole as made up of two factors, the original factor, which has already been thoroughly grasped, together with another unit. The proposition, "Three equals one, and one, and one," implies that each unit has been thought of singly, manifestly a more complex operation. It involves the conscious analysis of *two* into its elements.

If it should happen that the pupil at first begins to analyze the group into its simple elements, what inference could justly be drawn under such circumstances? Is it not simply that he is not in a position to think of *two* as a whole? Such procedure—thinking by ones—is conclusive evidence that he cannot think by *twos*, which is only another way of saying that he has not "mastered" *two*. Something has evidently been neglected in his previous consideration of "one of the parts which now make up this more complex whole." The parts have

not been "mastered as wholes." If a similar incident were to take place in connection with the study of geography, his teacher would have no manner of doubt as to the proper procedure. To revert to the illustration already used, if the pupil had been studying the theory of winds, and had been found wanting in his knowledge of gravitation, or of the expansive force of heat, he would have had no course open to him but to go back and "master, as wholes, the parts which now make up this more complex whole." The requirements of an exact science would seem to suggest the necessity of adhering to a course so eminently suited to the formation of clear concepts.

The first judgment, then, will be, "That three equals two and one." The pupil sees that two may be taken from three, and a remainder left. Employing the group he has already learned as a unit wherewith to measure the new group, he addresses himself to the question, "How often two may be taken from three?" and reaches the judgment: "In three there is one two and one over."

He is now in a position to proceed to the

measurement of the whole by the simple unit, and his judgment takes the form: "In three there are three ones," or "Three times one are three." This involves the following steps: "Three is two and one; two is two ones; therefore, three is three ones."

As already pointed out, our principle calls for the mastery of each portion or element of knowledge as a whole, and since this cannot be accomplished in the case of numbers without the explicit recognition of the ratio idea, we proceed to the study of the fractional relations implied in the idea of three. The attention of the pupil is therefore now directed to the equality of the units which go to make up the group he is considering. Here, for the second time in his experience in the scientific study of number, he finds the need of an expression to set forth the idea of ratio. He has already in his study of two reached the idea of one half. So here he gets the idea of one third and two thirds. These ideas arise in answer to the enquiries: "What part of three is one?" "What part of three is two?"

It may not be amiss to say at this point that

the pupil may not be in possession of the expression "one third" and "two thirds." What is stated is this: that having reached in his analysis of the group *three* the fact that three times one are three, and having next had his attention called to the equality of the units into which the whole has been divided, he can grasp the idea of the ratio between the unit and the group, and once in possession of the idea of the ratio no time need be lost in supplying him with the conventional symbol for that idea.

Is there any difference in point of complexity between these two propositions: In three, there are three ones, and, one third of three is one? In both we have the group, the unit, the number. In the former we have the group, *three;* the unit, *one;* the number *three;* in the latter the group *three,* the unit *one,* the number *one third.* The latter judgment is more complex than the former by the addition of the idea of the equality of the units. It is not until this element is brought into clear relief that the fraction idea is grasped. Does this question of the equality of the units present any difficulty to the mind of the child? In this

particular case the units are the simple units. Is not the equality of the simple units of the very essence of the numerical idea? It is true that some children do find difficulty at this point in regard to the equality or inequality of units. The cause is not far to seek. Such pupils it will be found are those whose idea of a unit is geometrical rather than arithmetical: who have been performing exercises in the measurement of extensive magnitudes, supposing that they were studying arithmetic. Children have been found who believed the number two to be red, the number three to be blue, and the number four to be square. As a matter of fact the child has no difficulty in getting from the judgment "In three there are three ones" to the judgment "one is one third of three." To take an illustration from applied arithmetic: suppose the child has successfully performed the operation involved in the following question—"A boy has two apples, to how many boys can he give one apple?" Is he likely to experience any serious difficulty in reaching a solution of the following problem: "If two apples are to be divided evenly between two boys, how many apples will one boy receive?" The first prob-

lem is a problem in division; the second is a problem in fractions, which is logically related to and grows out of the first.

3. There may be some advantage in further illustrating, by reference to a larger group, say, to six, the natural growth of numerical ideas in accordance with the principles which have been discussed and formulated.

Here again we may start with the assumption that the previous work has been mastered, that the pupil has clearly apprehended, as wholes, the parts which now go to make up this new whole. The problem we must remember, is that of measuring the new group by all its contained units. We shall bear in mind the fact that the line of least resistance is indicated by the degree of complexity of the judgments reached in this course of measurement, and that these judgments will form two classes—those in which the idea of ratio is merely implied, answering to the question, "how many?" (integer), and those in which the idea of ratio is fully expressed, answering to the question, "how much?" (a fraction).

If the pupil does not at first think of this

new group, six, as made up of the group last considered—five—and another unit, we are entitled on grounds already gone over to assume that the group five has not been sufficiently studied to admit of his grasping that idea in its totality. When six is thought of as five and one, we are in a position to proceed to the measurement of the new group by this most convenient unit. After finding what remainder is left when five is taken from six—clearly a less complex process than that involved in the problem how often five may be taken away—the statement "In six there is one five and one over" is reached. What now is required in order to measure our new group by the next unit, four? Obviously, six must now be regarded as "four and two"; and this is more complex than the former case since six must first be thought of as "five and one," then as "four and one and one," and then as "four and two." The idea that "In six there is one four and two left over" is more complex than the last mentioned one, and less so than the idea that "Six is made up of three and three."

It may, perhaps, appear, at first sight, that to

think of six as made up of three and three is simpler than to think of it as containing four once, with a remainder of two. If such is the case, we have missed the line of least resistance and lost our bearings. It is indeed maintained by some that an adequate idea of number can best be gained by a continued use of the same unit in measuring different quantities rather than by measurement by all its contained units, of one group after another in the order of their magnitude; and that the propositions "Three and three make six," "Two and two make four," "Two and two and two make six" stand for judgments which in the natural growth of numerical knowledge precede the analysis of the units themselves. It should however be tolerably clear by this time that the proposition "There are two threes in six" does not stand for any clear thought process in the mind of an individual who has not an adequate notion of the unit three. In order to avoid this error there are two important considerations to be kept clearly in view. One of these is to bear in mind that knowledge of any whole depends on knowledge of the parts which constitute the whole. The other is to understand the nature

of number. The idea that the proposition "In six there are two threes" is simpler than the proposition "In six there are four and two over" is probably due to a confusion between space and number. The division of part of a spatial whole, a line, or surface, or solid, into two equal parts may be "simpler" than its division into two unequal parts. If it were possible to reach an idea of number by a consideration of space alone, or if a spatial element were necessary to the formation of such idea, there would be some show of reason for this theory; but the former is not possible, and the latter is not necessary.

The problem of measuring by three is followed by measurement by two, resulting in the judgment, "There are three twos in six," and then by the final integral analysis, resulting in the judgment, "There are six ones in six."

Having now found answers to all the questions that have arisen regarding "*how many*" of a given unit in the whole, we now, in accordance with the principles laid down, proceed to deal with these units as "*so much*" of the whole. The result of considering the fractional relations involved in a number will be to build up the

idea of the number as a whole. When the last integral analysis has been reached, and the group thought of as composed of six simple units, the next step—the consideration of the equality of the units—takes us to a judgment explicitly setting forth the idea of ratio: "One is one sixth of six." This judgment involves the problem, "*How much* of the whole is one of these simple units?" That two sixths is a more complex idea than one sixth, and a less complex one than three sixths, four sixths, and five sixths, hardly needs any discussion. There will be no difficulty as to the order to be pursued in the further study of the group as a whole. It remains to be pointed out, however, that there are certain ratio ideas with which the pupil has already become acquainted, and of which use is made in the study of the number six. He became acquainted with the ideas of one half, one third, two thirds, while dealing with the numbers two and three. In order to grasp the truth that three is one half of six, or that two is one third of six, it is only necessary to make use of his formerly acquired ideas, by connecting them with the facts that in six there are two threes, and in six there are three twos.

4. The use of the phrase, "the line of least resistance," seems to imply a certain fixed order into which numerical ideas must necessarily fall, if there is logical control of one's thought-processes. A general order there certainly is, and it will be found that progress in numerical knowledge depends on adhering to it. Further, an order of steps may be made out in detail which should be rigidly followed in the study of the smaller numbers. As we come to the larger numbers, however, we shall find that the need of strict adherence in detail to one line of thought gradually relaxes, and greater variety becomes possible. Speaking generally, we may lay down the rule that, in the study of any given number, a consideration of the integral relations should precede that of the fractional ones. If, however, we compare certain judgments which involve new applications of fractional ideas already learned, with those which involve new integral relations, we find that there is not much difference between them in point of complexity. For example, in the consideration of the number six, the pupil at a certain stage reaches the judgment, "In six there are two threes." He has already become well acquainted with the idea of one half

in connection with his study of two and of four. At this point he would see at once that "Three is one half of six."

As ratio is of the very essence of number, we may conclude that the pupil who has a thorough grasp of the ratios implied in a number is in possession of an adequate idea of that number. An exercise in the comparison of ratios is the most searching test that can be applied. The mere act of comparing two ratios, as one third and two sixths, does not of itself require a great effort of intellectual power. The discovery or discernment of resemblance is the initial step in this quasi-inductive process. The equality of the ratios $1:3$ and $2:6$, however, can be observed only if there is a clear and definite idea of the ratios themselves. It is a commonplace to remark that the comparison of two ideas is rendered possible only by close acquaintance with the ideas themselves. On the other hand, the judgment, one third equals two sixths, or, more fully, $1 : 3$ as $2 : 6$, is a judgment easily reached by one who has a clear idea of the meaning of the judgment, "Two is one third of six," and the judgment, "Two is two sixths of six."

5. We have seen that the individual in giving order and system to what he encounters in the world of his experience employs among many other activities the power to number. It is to be borne in mind that the exercise of this power takes place without the intervention of the schoolmaster. It is not necessary to the healthy exercise of his spiritual powers that the child should call in the artificial and systematic aid of the teacher. The world of space and time is full of interest, and it is quite as natural that the objects which exist, and the events which happen therein, should be numbered, as that their size, colour, or duration should be observed and compared. We are not to suppose that the child on entering school is without any numerical ideas. Some teachers who are quite at home with the distinction between empirical and scientific in connection with knowledge of facts of botany or chemistry, are apt to overlook the fact that a foundation of numerical concepts has been laid in the mind of the child before he comes under the teacher's hands.

The child has become acquainted with the language of number in much the same way that

he has acquired the other parts of his vocabulary. Words like "some," "many," "few" are the first used. The want of something more definite is soon felt and supplied. It is probable, however, that prior to the employment of a unit for the more definite measurement of a vague multitude of things, the words one, two, three, etc., convey to the child the ordinal idea only. Numerical ideas he certainly has: though much of his knowledge is vague, much of his language inexact. It is for the teacher to increase his power to number by the judicious use of exercises and the correct use of conventional mathematical symbols.

The systematic study of the truths of number —the study of arithmetic as a science—as contemplated in accordance with the ideas expressed in these pages, cannot be carried on in any other way than by the unremitting use of the thought powers: by the formation of clear and distinct notions, by a careful and constant use of judgment and inference. It is believed that much of what is unsatisfactory in connection with the teaching of primary arithmetic is due to the unfortunate custom of attempting the scientific

study of the subject at too early an age. As a rule, children are not ready for the prosecution of a thought study, an exact science like arithmetic, at the age of five: and it is probable that no time is lost by postponing it till the age of seven has been reached.

6. Meantime the use of objects, *i.e.*, the employment of numbers of stones, spools, dots, musical notes, and what not, will be carried on, not according to any system more orderly than that of nature, but in the ordinary course of the child's daily experience. It will not be necessary for the teacher to supplement this experience by the "use of objects," as that phrase is generally understood by teachers. It should also be kept clearly in mind that exercises in "measuring"—the employment of the eye or the foot-rule for the comparison of quantities—are chiefly useful in the development of sense discrimination, not for that of numerical knowledge. We are not to lose sight of the importance of exercises which look to the development of sense perception, but it must be borne in mind that such exercises are in no way calculated to develop distinct ideas of number.

Long before the time has arrived for the scientific study of number, the child may and should acquire power to distinguish between the various kinds of space forms. This is the study of geometry. A child can discriminate between a triangle and a square without employing numerical ideas. It is true of course that he cannot express that difference in language without employing the language of number.

7. The advocates of the "use of objects" in teaching primary arithmetic attempt to justify the practice not only by an appeal to the authority of Grube, but also by the claim that mental arithmetic consists in forming mental pictures of objects corresponding to former actual experience in handling such objects. If this claim can be successfully established, the case of the advocates of the "objects" is a strong one. An illustration will show what, exactly, is meant by this. A problem is to be solved: "How many threes in seven?" The learner is asked to arrange his objects thus, ●●● ●●● ●; and when going through the "*mental*" exercises he is expected to call up before his mind's eye a picture corresponding to this. The picture

which he thus calls up differs from what he actually saw with his bodily eye only in point of vividness. The judgment he reaches in both cases as the result of his examination of the objects, actual and pictorial, is expressed thus: "In seven there are two threes and one over."

The mental processes, however, which are involved in performing arithmetical operations—whether mental arithmetic or any other kind of arithmetic—are thought processes. The processes which have just been described are not thought processes. The first is a perceptive process, and the second a representative one, and there is not any more thinking in the one case than in the other. We have noticed that a pictorial image is the intermediate step between perception and thinking in the formation of concepts (general notions): but "seven," as has been pointed out so often, is not a general notion. The true thought process by which the problem referred to is solved may be exhibited as follows: "Seven is six and one; six is made up of two threes; therefore in seven there are two threes and one over." Here the thought power is at work. We pass from one

judgment to another by a process of inference. So, in answering the question, "How many nines in sixty?" "In ten there is one nine and one over; in sixty there are six tens; therefore in sixty there are six nines and six over." It is difficult to see how the pictorial power can be brought into effective use in connection with large numbers; whereas the thought power is adequate to the performance of such processes even though very large numbers are employed. If arithmetic is properly a thought study, the pupil should be trained to think out numerical relations. The mere exercise of the perceptive and representative powers in connection with groups of objects is not arithmetic.

There is, after all, no reason why the pupil should handle blocks and spools in school. Before entering the school-room he has had a vast deal of experience in numbering objects and events, and is in possession of certain clear, though indistinct, ideas of number. He has not yet begun the study of the science of number, but he has made the necessary preparation for the study. And it may be remarked that the preparation referred to is quite as complete in

the case of a blind child as in the case of one who goes to the study of arithmetic fully equipped with all that is necessary for the formation of those pictorial images, which, in the theory we are combatting, are made to do duty for abstract thinking. If we could conceive a being deprived of the special senses of sight and touch—the senses by which we get our ideas of space and colour—such a being could certainly pursue the scientific study of arithmetic, the special sense of hearing providing the material; but he would be quite unable to solve any arithmetical problem by the employment of pictorial images.

8. Written exercises in arithmetic have a value of their own apart from the training in thinking out numerical relations, and that is the value which attaches to any exercise in the written expression of one's thoughts. Written exercises in arithmetic are, or should be, simply the expression of thought. The important difference between the two "kinds" of arithmetic is that in one case the thought is expressed orally, and in the other case in writing. It would seem at first sight as if for the purposes

of the teacher one form of expression would be as useful as the other. Such is not the case, however. While the pupil is engaged in the oral expression of thought, the teacher is in a position to test by means of questions the accuracy of the pupil's thinking. Not so while written work is being done by the class. The great danger in seat work is that the pupil may be acquiring evil habits, such as mechanical counting, the employment of the perceptive instead of the thought power, and, almost inevitably, the use of set forms of expression which save the pupil the trouble of thinking.

In connection with the teaching of arithmetic, careful attention is paid to the expression of the pupil's thought, not so much because the teacher wishes to cultivate the power of expression, but chiefly because he wishes to know exactly how and what the pupil is thinking. Expression here is scrutinized because it is the test of thought. Any expression which does not exhibit the process of thinking is of little value. As between the written and the oral expression of thought in the study of arithmetic, there is no hesitation in saying that the latter is far and

away the more valuable exercise. "Seat work" must be pronounced to be of very doubtful value. It is certainly liable to produce great evil to the thinking power of the children. The pupil gets into the way of setting down forms of expression which do not stand for real thought processes in his mind, but the use of which somehow enables him in a mysterious way to find the answer to the question. Such statements are not really the expression of the thought of the pupil. The figure of language as currency reminds us of the constant danger of counterfeit coin. Pupils who have been provided with a form in which they are to exhibit all solutions of problems do not necessarily acquire thereby either conciseness of expression or clearness of thought. They may, on the other hand, acquire the trick of appearing to perform mental processes which they do not really perform. The formal statement—

6 hats cost \$15,

$\therefore$ 1 hat will cost $\frac{1}{6}$ of \$15, hence

4 hats will cost 4 times $\frac{1}{6}$ of \$15 = \$10,

may be the expression of thinking by the pupils, or it may be a "blank cheque of intellectual

bankruptcy." A "right answer" may always and easily be found by rule if the pupil only knows how to do the trick.

The problem is evidently a problem in proportion. The pupil who really solves the problem must know that the number required is the number which stands in the same relation to 15 that 4 does to 6. He applies his knowledge of pure number to a concrete case. He knows that the ratio 10 : 15 is always equal to the ratio 4 : 6. If there is any advantage to the pupil in writing down the entire chain of reasoning in a complicated problem that advantage will be made secure by insisting on plenty of oral expression before the pupil is asked to write. It is not too much to say that nine-tenths of the difficulty experienced by pupils in the solution of problems would disappear if the terms used in stating them were thoroughly understood. Here again is the weakness of written work. Oral questioning would force the pupil to think out the solution, and the difficulty would disappear, because it would be impossible then to employ terms which were not thoroughly understood.

Much has been said about the difficulty of providing suitable employment at seats for children of the elementary grades. For a long time a principal source of supply for exercises at seat has been the "number work"; and if this source is cut off the difficulty for some teachers will be greater than ever. The problem, however, is not to keep the children employed, but how to keep them profitably and happily employed. Arithmetic is an important subject, no doubt, and must always occupy a considerable place on our school programmes. But it should not be allowed to permeate every department of school study, and dominate every hour of the school day. It is not an overstatement of fact to affirm that there are many teachers who try to keep the children employed in the study of number during one-third of their time in the school-room, in class and at seats. There is a double blunder in these cases. Teachers who so arrange their time tables should first revise their ideas regarding education values; and then, when they have reduced arithmetic to its proper rank as a school study, try to form a correct estimate of the value of "written" arithmetic.

9. It may be added that much of the "*useful*" material which has been crowded into the text books, though called by the name of arithmetic is not arithmetic at all. A knowledge of the usages of the shop and market-place, the number of cents in a sovereign, how to write a non-negotiable promissory note, the buying and selling of bonds, etc., all these things are both interesting and useful, but they are not arithmetic, do not belong to the science. To solve an intricate problem in stocks is looked upon as a creditable feat for a school-boy. It is; but the difficult part of the feat is not in the arithmetical processes performed, but in understanding the ins and outs of the business of the broker. There are some teachers who would consider the following questions as quite difficult questions in arithmetic: How many bushels in a ton of bran? How many pyramids the size of the great pyramid of Egypt would it take to cover a half-breed claim on the Red River? Find the weight of the gas in Andree's balloon. Reduce three York shillings to Egyptian currency. The difficulty, wherever it is, is not in the arithmetical processes necessary to answer these questions.

10. The system of notation which we use is called the decimal system, because ten is the radix or basis. If twelve were adopted as the base instead of ten, it is clear that there would be a great gain in one important respect, namely, in the fact that whereas the latter has only two divisors, the former has four. The foot of twelve inches, the pound of twelve ounces, the shilling of twelve pence, the year of twelve months, the day of twelve hours, the dozen, the gross, and great gross, have hitherto resisted the effort to secure complete conformity with the decimal system. The usual and obvious way of accounting for the universal adoption of ten as the basis of numerical calculation and notation is by reference to the ten fingers of the human hand.

When the number ten has been mastered, the learner has reached a very important point in his career as a student of the science of arithmetic: not, of course, on account of anything magical in connection with the number ten, but simply because ten has by agreement been fixed as the basis of our system of notation. The science of arithmetic should present but few

difficulties from this point onward. The learner is now in possession of knowledge which will enable him to solve any question that can possibly arise in the domain of pure number. He is in possession of an exhaustive knowledge of ten, which has been well designated "the key to number."

If the procedure marked out by the dictum on page 55 has not been followed, if the pupil's knowledge is vague or incomplete through his failure to "master as wholes the parts which now make up this more complex whole," he will now more than ever be hampered in his progress. The problem to which he has now to address himself is that of becoming acquainted with our system of notation. The numbers greater than ten present nothing that has not been practically mastered. Eleven means *ten* and one, fifteen means *ten* and five, twenty means two *tens*, forty-five means four *tens* and five. The pupil whose study of arithmetic has been a scientific study will now go forward by leaps and bounds.

The manifest duty of the teacher at this stage is to show the pupil that he holds the key

which is to unlock all future problems in number. Exercises are now to be provided which shall give the learner the consciousness of the power he possesses. He must form the habit of computing by tens.

A few examples will suffice to show the method to be followed. The answers to the questions (*a*) $16 \div 5$; (*b*) $\frac{4}{5}$ of 20; (*c*) $30 \div 8$ may be reached in either of two ways.

I. (*a*) $5+5=10$; $10+5=15$; $\therefore$ in 16 there are 3 fives and 1 over.

(*b*) $\frac{1}{5}$ of $20=4$; $\therefore$ $\frac{4}{5}$ of $20=4$ times $4=16$

(*c*) $8+8=16$; $16+8=24$; $30-24=6$; $\therefore$ in 30 there are 3 eights and 6 over.

II. (*a*) $16=10+6$; in ten there are 2 fives; in 6 there is one 5 and 1 over; $\therefore$ in 16 there are 3 fives and 1 over.

(*b*) $\frac{4}{5}$ of 10 is 8; there are 2 tens in 20; $\therefore$ $\frac{4}{5}$ of 20 is twice 8 or 16.

(*c*) In 10 there is one 8 and 2 over; in 30, or 3 tens, there are 3 eights and 6 over.

The latter of these modes is preferable in that it explicitly recognizes the decimal system; it tends to establish the habit of analysis of large

groups into tens; and it makes perfectly clear all that needs to be learned at this stage regarding the conventional method of notation. The former plan is faulty in that it does not do any of these things. It may further be remarked that in the second case there is a more systematic use made of old truth in the discovery of new truth, than in the first.

Sufficient practice in exercises of this kind must be given in order to make quite clear to the mind of the pupil the fundamental fact that numbers greater than ten are expressed in terms of ten, and must be so thought of and treated in numerical calculation. If this point is clearly recognized in all the work done at this stage, there will be no difficulty at a later stage in understanding that "every figure placed to the left of another represents ten times as much as if it were in the place of that other."

CHAPTER V.

EXERCISES.

1. The Number Six.—(A) Pure Arithmetic.

(*a*) 1. What number with five makes six ?

2. What number with one makes six ?

3. Six, take away five, leaves what ?

4. Six, take away one, leaves what ?

5. How many times can we take five away from six ?

6. One five and one more make how many ?

(*b*) 1. What number with four makes six ?

2. What number with two makes six ?

3. Six, take away four, leaves how many ?

4. Six, take away two, leaves how many ?

5. How often can we take four away from six ?

6. One four and two make how many ?

(*c*) 1. What goes with three to make six ?

2. Six, take away three, leaves how many ?

3. How often can we take three away from six ?

4. Two threes (or twice three) are how many ?

(*d*) 1. What goes with two to make six ?

2. How often can we take two away from six ?

3. Three times two make how many ?

(*e*) 1. What goes with one to make six ?

2. How many times can you take one away from six ?

3. Six is how many times one ?

(*f*) 1. When six is divided into ones, it is divided into how many equal parts ? (Give name, if necessary.)

2. Two is what part of six ?

3. Two is what other part of six ? What is the same part of six that one is of three ?

4. Three is what part of six ?

5. Three is what other part of six ? What number is the same part of six that one is of two ? or that two is of four ?

6. Four is what part of six ?

7. Four is what other part of six? What number is the same part of six that two is of three?

8. Five is what part of six?

9. One half of six and one third of six are how many?

10. Two thirds of six, less one half of six, are how many? etc.

(B) Applied Arithmetic.

(*a*) 1. An exercise book costs six cents. John has only five cents. How much more money must he get in order to buy it?

2. A boy has six marbles. There is one in one pocket. How many in the other?

3. A boy has to carry six armfuls of wood. He has already carried five. How many more has he to carry?

4. It is six o'clock. The clock has struck once. How many times has it yet to strike?

5. There are six pupils in the school. How many classes of five can the teacher make?

6. Pencils are sold in bundles of five. A boy wants six pencils.. How many bundles must he get?

(*b*) 1. A pole is six feet high. A spider has crept up four feet. How far has he yet to go?

2. There are six calves in the stable. Two of them are lying down. How many are standing?

3. I have six silver pieces. There are four of them in a dollar. How many dollars have I?

4. There are four farthings in a penny. How many farthings are there in one penny and two farthings?

(*c*) 1. You have been promised six plums, and have already received three. How many are left?

2. There are six boys in a room. Three go out? How many are left?

3. Six men wish to cross a river. The boat will only carry three passengers. How many trips must be made?

4. It takes three horses to draw the binder. There are two binders at work in the field. How many horses?

(*d*) 1. A man has six cows. Two of them are in the stable. How many are outside?

2. There are six horses, and two can stand in each stall. How many stalls do you need?

3. Find the cost of three two-cent stamps.

(*e*) 1. There are six freight cars on the track. Only one is loaded. How many are empty?

2. I have six cartridges. How many chickens can I shoot with them if I shoot one with each cartridge?

3. How many apples can you buy at one cent each if you have six cents?

(*f*) 1. Six oranges are to be divided among six boys. How many will one boy get.

2. What part of the oranges does one boy get?

3. There are six working days in a week. A boy works Monday and Tuesday. What part of the week has he worked?

4. If he is to get three dollars a week, what will he get for the two days' work ?
5. Six books cost two dollars. What will three books cost ? How many can I buy with one dollar ?
6. It is six miles from the school to the post office. A boy has walked four miles. What part of the distance has he walked ?
7. A man has worked five sixths of the week. How many days has he worked ?

2. The Number Ten.—(A) Pure Arithmetic.

(*a*) 1. What number with nine makes ten ?
2. What number with one makes ten ?
3. Ten, take away nine, leaves what ?
4. Ten, take away one, leaves what ?
5. How many times can we take nine away from ten ?
6. One nine and one make how many ?

(*b*) 1. What number with eight makes ten ?
2. What number with two makes ten ?
3. Ten, take away eight, leaves how many ?
4. Ten, take away two, leaves how many ?

5. How many eights in ten?

6. One eight and two make how many?

(*c*) 1. What number with seven makes ten?

2. What number with three makes ten?

3. Ten, take away seven, leaves what?

4. Ten, take away three, leaves what?

5. Ten is how many sevens?

6. One seven and three make how many?

(*d*) 1. Six, with what other number makes ten?

2. Four, with what number makes ten?

3. Ten, less six, leaves what?

4. Ten, less four, leaves what?

5. How many sixes in ten?

6. One six and four ones make what?

(*e*) 1. Five, with what other number makes ten?

2. Ten, take away five, leaves what?

3. How many fives in ten?

4. Two fives make what?

(*f*) 1. Four with what number makes ten?

2. Ten, take away four, leaves what?

3. How many fours in ten?

4. Two fours and two ones make what?

(*g*) 1. What number goes with three to make ten?

2. Ten is how many threes?

3. Three threes and one are how many?

(*h*) 1. Two and what number makes ten?

2. Ten is how many twos?

3. Five twos make how many?

(*i*) 1. One and how many makes ten?

2. Ten is how many ones?

3. Ten times one is what number?

(*j*) 1. Think ten into ones. How many equal parts are there?

2. What shall we call one of these parts?

3. What shall we call two of these parts, three of these parts, four of these parts? etc.

4. Think ten into twos. How many equal parts are there? Therefore two is what part of ten? Compare one fifth of ten with two tenths of ten.

5. Compare two fifths of ten with four tenths of ten, three fifths of ten with six tenths of ten, etc.

(B) Applied Arithmetic.

(*a*) 1. There are ten pupils in a rural school. There are nine boys. How many girls are there?

2. There are ten pupils in two classes. In one of the classes there is only one pupil. How many pupils are there in the other?

3. A boy has ten cents. He spent nine cents. How many cents has he left?

4. John has ten cents. How often can he give his brother nine cents?

5. There are nine sheaves in a stook. One stook and one sheaf makes how many?

(*b*) 1. It is ten miles from Brandon to Harrow. A man has ridden eight miles. How far has he yet to go?

2. Ten pigeons on the roof. Two have flown. How many are left?

3. How many lead pencils at eight cents each can be bought with ten cents?

4. Johnny can carry eight bricks in his little hod. Mary carries two in her hands. How many bricks have they?

(*c*) 1. A boy wishes to buy a ten cent scribbler. He has seven cents. How much more must he get?

2. A hen has ten chickens. A hawk takes three. How many are left?

3. John has ten sheep. He can put seven in each stable. How many stables are needed?

4. How many days in one week and three days?

(*d*) 1. Henry has ten marbles in two pockets. He has six in one pocket. How many in the other?

2. Mary has ten cents. She spends four cents. How many cents has she left?

3. There are six days in a working week. A man has worked ten days. How many weeks has he worked?

4. One working week with four days is how many days?

(*e*) 1. A boy has ten cents. He loses five. What is left?

2. How many exercise books at five cents each can be bought for ten cents?

3. Two lead pencils at five cents each would cost how much?

(*f*) 1. There are four farthings in a penny. How many pence in ten farthings?

2. A man has two four horse teams, and two drivers. How many horses has he?

(*g*) 1. John has ten matches. How many triangles can he make?

2. On a farm three horses are required for each of three binders, and one as a carriage horse. How many horses in all?

(*h*) 1. How many double seats will be required for ten boys?

2. Find the cost of five two-cent stamps.

(*i*) 1. At one cent each, how many slate pencils can be bought for ten cents?

2. What is the cost of ten alleys at one cent each?

(*j*) 1. A hen has ten chickens. She loses one. What part of her flock has she lost? What part has she left?

2. A boy starts to ride ten miles. At the end of the eighth mile he punctures his tire, and walks the remainder. What part of the distance does he walk? What part has he ridden?

3. Mary has ten cents. She spends five cents for an exercise book. What part of her money has she spent?

3. Exercises Emphasizing "Ten" as the Base for all Arithmetical Operations.

(*a*) 1. What number with ten makes twenty?

2. Twenty, take away ten, leaves what?

3. How many tens in twenty?

4. Two tens make what?

5. In twenty, how many nines?

(*Solution:*

First step—Think twenty into tens.
Second step—Think each ten into nines.
Third step—Combine nines and ones.)

6. In twenty, how many eights?

7. In twenty, how many sevens?

8. In twenty, how many sixes?

9. In twenty how many fives?

10. In twenty, how many fours?

11. In twenty, how many threes?

12. In twenty, how many twos?

13. In twenty, how many ones?

14. What is one tenth of twenty?

(*Solution:*

$\frac{1}{10}$ of 10 is 1, $\therefore$ $\frac{1}{10}$ of 20 is twice $1=2$.)

15. What is one fifth of twenty?

16. What is two fifths of twenty?

17. What is seven tenths of twenty?

(*b*) 1. What number with twenty makes thirty?

2. Thirty, take away ten, leaves what?

3. Thirty, take away two tens, leaves what?

4. In thirty, how many twenties?

5. How many tens in thirty?

6. Three tens make what?

7. In thirty, how many nines?

8. In thirty, how many eights?

9. In thirty, how many sevens?

4. Notation.

1. From two tens take away nine ones.

(*a*) How many whole tens are left?

(*b*) How many separate ones are left?

(*c*) What does the term *eleven* mean exactly?

2. From two tens take away one ten and one. How many are left?

3. From two tens take away eight ones. How many tens and how many ones are left? What does the term *twelve* mean?

4. From two tens take away one ten and two ones? How many are left?

5. From twenty take away seven. How many are left? What does the term *thirteen* mean?

Proceed similarly with the others.

5. MISCELLANEOUS EXERCISES.

1. How many exercise books worth ten cents each can be bought for twenty cents?

2. A boy has twenty cents. He spends five cents for a lead pencil. How many copy books at five cents each can he buy with the remainder?

3. It is twenty miles from Boissevain to Killarney. A man rides ten miles of the distance to-day. How many miles has he still to go, and what part of the distance has he travelled?

4. There are nine square feet in a square yard. How many square yards in twenty square feet?

5. There are three feet in a yard. How many yards in twenty feet?

6. There are twelve inches in a foot. How many feet in twenty inches?

7. There are sixteen ounces in a pound. How many pounds in twenty ounces?

8. What part of a yard is a foot?

9. What part of a foot is one inch? three inches? five inches? eight inches?

10. There are twelve ounces in a pound Troy. How many pounds Troy in twenty ounces?

11. There are twenty shillings in a pound. What part of a pound is fifteen shillings?

12. How many pounds of rice at eight cents can be bought for twenty cents?

13. How many weeks in twenty days?

14. How many days in two weeks and six days?

15. There are six working days in a week. How many working weeks in twenty days?

16. A plot of ground is four feet by five feet How many square feet does it contain?

17. Five square feet of the plot is in grass. What part of the whole is in grass?

18. How many pounds of oatmeal at four cents a pound can be bought for twenty cents?

19. Lawn is twenty cents a yard. Find the cost of half a yard—of a quarter of a yard?

20. Four yards of cotton cost twenty cents. What will three yards cost?

21. Six pencils cost eight cents. What will three pencils cost? What will two cost?

22. Six hats cost nine dollars. What will four hats cost?

23. John has two cents. James has one cent. They put their money together and buy eighteen marbles. They then divide them fairly. How many should each get?

24. John had ten cents. He spent two cents. What part of his money did he spend? What part has he left?

25. A boy caught ten fish, and gave away two fifths. How many had he left?

26. John has twelve cents. He spent one third of his money for an exercise book, and one half of it for a lead pencil. What part of his money did he spend altogether? The difference between what he spent for the pencil and what he spent for the book is what part of his money?

27. A boy has ten cents. He spends one fifth in marbles and one half of the remainder for an exercise book. What part of his money did he spend for the exercise book? What part did he spend altogether? What part has he left?

28. Eight apples cost twelve cents. What will six apples cost?

29. How many apples will three cents buy?

30. One yard of tape costs sixteen cents. What will nine inches cost?

31. There are eight gallons in a bushel. One bushel of oats costs twenty cents. What will six gallons cost?

32. Oats are sixteen cents a bushel. Find cost of seven gallons.

33. Eggs are eighteen cents a dozen. How many eggs can you get for ten cents?

34. Oranges are sold at the rate of eight for twenty cents. A boy has fifteen cents. How many can he buy?

35. John caught two fish. James caught three. They sold their fish for fifteen cents. How did they divide their money?

36. A man had three horses. His brother had four. They hired them out together, and received fourteen dollars. How did they divide their money?

37. Divide fifteen cents between John and James so that John will have twice as much as James.

38. Of twelve calves, five are black, one third are white, and the rest are red. What part are red?

39. John and Mary have ten plums each. John eats six tenths of his, and Mary eats three fifths of hers. How many has each now?

40. Three sheep are exchanged for five pigs. At the same rate how many sheep would you get for fifteen pigs?

41. Ten men can stack a field of grain in three days. How many men would it take to do the work in five days?

42. Divide eighteen apples between John and May, so that every time John gets four May will get five.

43. A boy had twelve cents. He spent one third of it in candy, and one fourth of the remainder in popcorn. How many cents had he left? What part of the whole had he left?

44. Two fifths of a post is in the ground, and there are nine feet of it above ground. How long is the post?

45. John has sixteen cents. He spends three eighths of it and loses one fourth. How many eents has he left? What part of the whole?

46. A man sold two sheep at twelve dollars eaeh. On one he gained one third and on the other he lost one third of what he paid. Did he gain or lose, and how much?

47. Two apples and two oranges cost fifteen cents. Two oranges cost as much as three apples. Find the cost of an apple.

48. John has five cents. Dick has three. They put their money together and buy sixteen apples. How should they divide the apples?

49. Seven men do a piece of work in sixteen days. How long would it take eight men?

50. How many men would do it in seven days?

51. John can wheel three times as fast as James can walk. They start off in opposite directions, and stop when they are twenty rods apart. How far has James walked?

52. In returning, James runs two thirds as fast as John rides. Where will they meet?

53. How many ounces in one pound and one third Troy?

54. What is the difference between two thirds of a pound and three fourths of a pound? What part of a pound is it?

55. John can walk four miles an hour. How long will it take him to travel eighteen miles ?

56. How far will he walk in two hours and a half ?

57. How far will he walk in four hours and three quarters ?

58. Rice is six cents a pound. Find the cost of two pounds and one third.

59. Find the cost of one pound and two thirds.

60. John has six cents. William has four. They club together and buy newspapers. They sell, and gain a profit of twelve cents. How will they divide their profit ?

61. James has eighteen sheep, and sells fourteen. What part did he sell ? What part is left ?

62. There are ten hours in a working day. Wages are five dollars a day. What would a man earn who worked two hours a day for three days ?

63. There are twenty calves. Each stall will hold three calves. How many stalls will be used ?

64. What part of the whole number will be in the last stall?

65. What part will be in the other stalls?

66. Three men own a ship. One owns one third, another two fifths. What part does the third man own?

67. A boy has eighteen cents. He buys five oranges at three cents each. What part of his money has he spent?

68. There are five classes in a school of eighteen pupils. There are two pupils in the highest class, and the other pupils are equally divided among the other classes. What part of the whole are in one of the other classes? in three of the other classes?

69. It takes John six minutes to write out sixteen words. How long will it take him to write out twelve words?

70. There are fifteen minutes allowed for recess. John is kept in twelve minutes, and Mary ten minutes. What part of the usual time has each left for play?

71. Fifteen men do a work in twelve days. How many men could do it in twenty days?

72. A railway train goes twenty miles an hour. How far would it go in three fifths of an hour?

73. A man has eighteen horses. Two thirds are grey, two ninths are black, and the remainder white. What part are white? What part are not black?

74. Charlie had sixteen marbles. He gave three eighths to Tom, and one fourth to James. What part had he left? How many more had Tom than James? How much more?

75. Divide eighteen cents between two boys, in the proportion of four to five.

76. Divide twenty cents among three boys, in the proportion of two, three, and five.

It is unnecessary to suggest further exercises, as all other numbers are aggregations of tens and parts of ten.